EPICYCLIC GEARING

KINEMATICS & TORQUE DISTRIBUTION

By

W. BROWN

MACHINERY'S
YELLOW BACK
— SERIES —

No. 33

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PREFACE

Although most text-books on gearing devote some space to epicyclic gearing, and more specialized articles appear from time to time in the engineering press, there is still a lack of collected information on the kinematics of epicyclic gears in general. Text-books usually confine themselves to the analysis of simple trains and the lower-order coupled trains, whilst the specialized articles tend to over-elaboration. There is, moreover, a paucity of information on the synthesis of trains from first principles.

It is the purpose of this book to remedy, in some measure at least, the defects in epicyclic literature and to do so in as straightforward a manner as possible. With these ends in view the notation has been kept as simple and self-explanatory as possible. The analysis of existing trains has been fully dealt with to familiarize the reader with the basic principles of epicyclic gears and the synthesis of trains to meet specific requirements is discussed, both generally and in the form of examples.

No excuse is made for the fact that some of the material is old and well-known: epicyclic gears also are old and well-known up to a point. Such material as is old should be looked upon as the bricks from which the book is built and such material as is new should be looked upon as the mortar which connects the bricks so as to produce a unified whole. The proportion of mortar to bricks will depend upon the past experience of the reader. The writer trusts that there will be more than mere bricks for the experienced and that the mortar will not be too indigestible for the inexperienced.

The approach to the subject is a practical one, free from abstract symbolism, and none of the calculations involved require more mathematical ability than the manipulation of algebraic expressions of the first degree.

Some of the material used in Chapters 1 and 2, and that used in Chapter 4, has been drawn in somewhat modified form from an article by the author which appeared in *Practical Engineering*, and grateful acknowledgement is made to that journal for permission to use it.

W. B.

CONTENTS

CHAPTER 1

THE BASIC TRAIN

1.1 Since all epicyclic gears, however complex, are derived from the basic train of sun, pinion, planet carrier, planet pinions and annulus, a complete understanding of the relative movements of these elements is essential before commencing the study of the more complex trains. Fig 1 shows the basic train and it is convenient to use the letters S, P, C and A to designate the sun, planets, carrier and annulus respectively. In the present book, S, P and A are used to denote the numbers of teeth in the toothed members and the letters s, p and a to denote the rotational speeds of the same members, c being the rotational speed of the planet carrier. In all cases the sign (+) or (−) must be inserted to show the sense of rotation, since, in many of the ratios discussed later, reverse gears are obtained. Consideration of the train in Fig 1 shows that there are three coaxial members and that three interdependent rotational speeds can occur simultaneously. To reduce this to two interdependent speeds for which a ratio can be quoted, it is necessary to apply a restraint to one of the members, normally by fixing the chosen member against rotation. Alternatively, two of the members may be driven at known speeds so that the speed of the third may be given, or any one of the members may be driven from any other at a known ratio which again allows the absolute speed of the third to be stated.

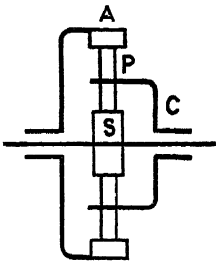


Fig 1. The Elements of the Basic Epicyclic Train showing the Symbols adopted throughout the Book

It is this simultaneous rotation of three members which gives rise to the feeling that epicyclic gears are difficult to understand, particularly where two or more members are coupled together. It is for this reason that a complete grasp of the functioning of the basic train is so important.

1.2 To establish the relative movements of the train in Fig 1, first consider all the elements to be rotated *en bloc* through +1 revolution

and then, with the planet carrier fixed against rotation, let the sun be rotated —1 revolution. The relative movements of the elements so obtained are tabulated and added as follows:

Revs of Sun	Revs of Planet	Revs of Carrier	Revs of Annulus
$+1$	$+1$	$+1$	$+1$
-1	$+S/P$	0	$+S/A$
0	$+\dfrac{P+S}{P}$	$+1$	$+\dfrac{A+S}{A}$

The second line gives the relative movements of sun, planet and annulus when the carrier is fixed, whilst the third line gives the relative movements of pinion, carrier and annulus when the sun is fixed. Since all three of the members may be rotating, it is necessary to multiply the second line by $-s$ and the third line by c, again tabulating and adding so as to obtain the combined relative speeds, thus:

Revs of Sun	Revs of Planet	Revs of Carrier	Revs of Annulus
Carrier Fixed:			
$+s$	$-\dfrac{sS}{P}$	0	$-\dfrac{sS}{A}$
Sun Fixed:			
0	$+\dfrac{c(P+S)}{P}$	$+c$	$+\dfrac{c(A+S)}{A}$
Combined:			
$+s$	$+\dfrac{c(P+S)-sS}{P}$	$+c$	$+\dfrac{c(A+S)-sS}{A}$

This gives the rotation speed of the planet as

$p = \dfrac{c\,(P+S) - sS}{P}$; that is,

$$pP + sS = c\,(P + S) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (1)$$

and, $a = \dfrac{c\,(A+S) - sS}{A}$, whence

$$aA + sS = c\,(A + S) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (2)$$

These two equations are sufficient to define the relative movements of all four members of the basic train. Equation (2) is the characteristic equation for the three coaxial members. Where two or more

trains are coupled together, however, it is sometimes necessary to know the movements of the planets when some of the members are missing. Two more equations are required and these are obtained as follows:

Combining (1) and (2) to eliminate s and S we have:

$aA - pP = c(A+S) - c(P+S)$, hence

$$aA - pP = c(A - P) \quad \ldots \quad (3)$$

Examination of the basic train shows that $A - P = P + S$ and $A - S = 2P$. Using this with equations (1) and (3) gives:

$pP + sS = c(P + S) = c(A - P) = aA - pP$; therefore

$$aA - sS = 2pP = p(A - S) \quad \ldots \quad (4)$$

Equations (1) to (4) cover all the possible relative movements of epicyclic gear trains, including the basic train, two or more coupled trains, or coupled trains in which one or more members may be missing.

1.3 In applying the four equations to the analysis of the basic train it will be observed that, as far as the coaxial members are concerned, only equation (2) is required. If a reaction member is chosen, then the terms containing the speed of this member are reduced to zero, giving:

S stationary: $aA = c(A + S) \quad \ldots \quad (5)$

C stationary: $aA + sS = 0 \quad \ldots \quad (6)$

A stationary: $sS = c(A + S) \quad \ldots \quad (7)$

Three ratios are thus obtainable from the basic train and these can all be expressed in terms of A and S. It is sometimes said that there are six ratios, but this is misleading as the other three are only inversions of the three already given.

As a practical example, take a basic train with a 15-tooth sun pinion, 21-tooth planets and a 57-tooth annulus and assume the sun to be the reaction member. The speed ratio of annulus to cage is required. From equation (5), when $s = 0$, $aA = c(A + S)$

$\therefore\ a \times 57 = c \times 72$ giving $\frac{a}{c} = \frac{24}{19}$.

CHAPTER 2

COUPLED TRAINS (2nd Order)

2.1 The conditions for the least coupling between two basic trains to give an absolute ratio between one member of one train and one member of the other train are that two members of one train must be coupled to two members of the other train and that one reaction member is provided. This reaction member may be one of the coupled pairs. Further, although there are four members present in each basic train, it is generally only possible to choose the coupling members from the three coaxial ones. The only exception to this is when the two cages are coupled. It is then possible, by making the cage common to the two trains, to couple the planet pinions. This can be done by using so-called " compound " pinions (two diameters) or by meshing one set of planet pinions with the other. (*See* 2.6.)

2.2 Chart I shows in the diagrams (*a*) to (*i*) all the possible ways of coupling two basic trains, the thick line in each case being one of the couplings and the thin lines being the alternative couplings available. The small figures on the thin lines refer to Chart II. It will be seen that, generally, there are four alternative couplings, but in the case where the cage is common six alternative couplings are available.

2.3 Chart II shows the practical couplings which, due to " mirror " cases, comprise only fourteen practical cases. The selection chart in the right-hand bottom corner shows that, whilst there should be eighteen normal cases, there are six " mirror " cases, in addition to which there are the two cases of common cage and coupled planets, one being " compound " and one " meshing."

2.4 It was shown in 1.3 that the basic train gives three ratios, *i.e.* as many ratios as there are coaxial members. Now, in the case of 2nd-order trains, Chart II shows that in twelve cases there are four coaxial members and in two cases, five. Where four coaxial members are present, a choice of four reaction members is available with a further choice of any two from the remaining three as input and

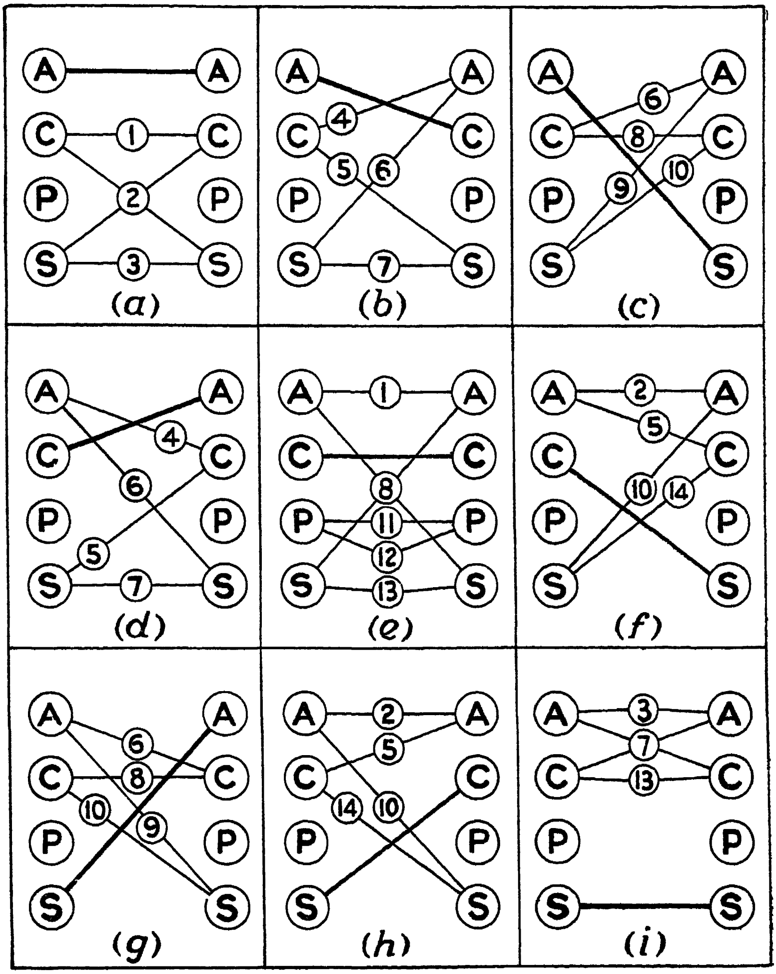
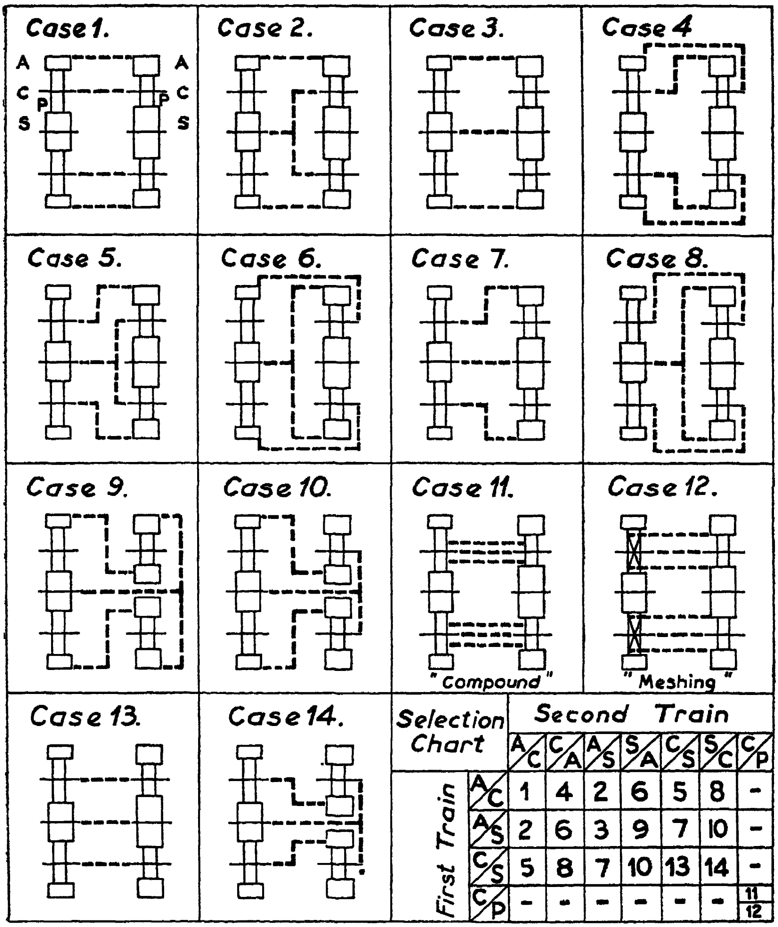


CHART I

output members. There are thus $4 \times 3 = 12$ ratios available in these cases. Where five coaxial members are present, the ratios available are $5 \times 6 = 30$. The total number of ratios available from two coupled trains from which choice can be made is, therefore, $12 \times 12 + 2 \times 30 = 204$.



Selection Chart: First Train \ Second Train	A/C	C/A	A/S	S/A	C/S	S/C	C/P
A/C	1	4	2	6	5	8	-
A/S	2	6	3	9	7	10	-
C/S	5	8	7	10	13	14	-
C/P	-	-	-	-	-	-	11/12

CHART II

2.5 Velocity Equations

Whilst it would be possible to write down the velocity equations for the whole of the 204 ratios, it is more practical to show the derivation of one particular example, using equations (1) to (4) from 1.2. Select, say, Case 8 (Chart II) and let S_2 be the reaction member, C_2 be the input member and A_1 be the output member. The member C_1 is coupled to C_2, S_1 to A_2 and the ratio $\frac{a_1}{c_1}$ is required.

From (2) $a_1 A_1 + s_1 S_1 = c_1 (A_1 + S_1)$ ... ... (8)

and $a_2 A_2 + s_2 S_2 = c_2 (A_2 + S_2)$ ... ... (9)

Since $s_2 = 0$, $c_1 = c_2$ and $s_1 = a_2$ these equations can be rewritten as

$$a_1 A_1 + a_2 S_1 = c_2 (A_1 + S_1) \quad \ldots \quad (10)$$

$$\text{and } a_2 A_2 + 0 = c_2 (A_2 + S_2) \quad \ldots \quad (11)$$

from (11) $a_2 = c_2 \left(\frac{A_2 + S_2}{A_2}\right)$. Substituting in (10) gives:

$$a_1 A_1 + c_2 \left(\frac{A_2 + S_2}{A_2}\right) S_1 = c_2 (A_1 + S_1).$$

Hence,

$$\begin{aligned} a_1 A_1 &= c_2 (A_1 + S_1) - c_2 \left(\frac{A_2 + S_2}{A_2}\right) S_1 \\ &= c_2 \left\{ (A_1 + S_1) - \left(\frac{A_2 + S_2}{A_2}\right) S_1 \right\} \\ &= c_2 \left\{ \frac{A_2 (A_1 + S_1) - (A_2 + S_2) S_1}{A_2} \right\} \\ &= c_2 \left\{ \frac{A_2 A_1 + A_2 S_1 - A_2 S_1 - S_2 S_1}{A_2} \right\} \end{aligned}$$

and,

$$\frac{a_1}{c_2} = \frac{A_2 A_1 - S_2 S_1}{A_2 A_1} \quad \ldots \quad (12)$$

The working of this example has been shown in full because facility in the manipulation of the four equations is essential when synthesising trains, as will be shown later. When analysing an existing train it is sometimes simpler to use the method shown in 1.2, as follows: Write down the essential elements, consider them to be given + 1 revolution *en bloc* and rotate the reaction member − 1 revolution whilst holding the carrier stationary. The resulting movements are tabulated and added.

(S_1)	(C_1)	(A_1)	(S_2)	(C_2)	(A_2)
$+1$	$+1$	$+1$	$+1$	$+1$	$+1$
$+\frac{S_2}{A_2}$	0	$-\frac{S_2}{A_2}\frac{S_1}{A_1}$	-1	0	$+\frac{S_2}{A_2}$
$+\frac{A_2 + S_2}{A_2}$	$+1$	$\frac{A_2 A_1 - S_2 S_1}{A_2 A_1}$	0	$+1$	$+\frac{A_2 + S_2}{A_2}$

This gives $\frac{a_1}{c_2} = \frac{A_2 A_1 - S_2 S_1}{A_2 A_1}$ which is the same as (12) above.

This example was chosen deliberately because the cages are coupled. When this is so, the tabulating method is simpler than using the equations. When the cages are not coupled it is generally preferable to use the equations, or to use both methods together.

2.6 Compound Trains. (*See* 2.1.)

Trains using "compound," or stepped, planet pinions are usually considered as a special case. Such a train is shown in Fig 2. Examination of this train will show that this is a coupled train of the type shown as Case 11, Chart II.

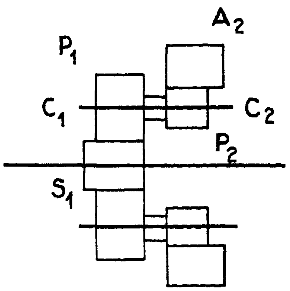


Fig 2. A Train with Stepped or Compound Pinions

The annulus of the first train and the sun of the second train are missing, but this does not affect the issue, the train being a 2nd-order train with the double coupling of C_1 to C_2 and P_1 to P_2. The velocity equations can be obtained by the tabulating method shown in 1.2 or by use of the equations.

From (1) and (2) and the fact that $p_1 = p_2$ and $c_1 = c_2$ we have

$$p_2\, P_1 + s_1\, S_1 = c_2\,(P_1 + S_1) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

and $a_2\, A_2 - p_2\, P_2 = c_2\,(A_2 - P_2)$ ... ... (14)

rearranging and dividing gives

$$\frac{p_2\, P_1}{p_2\, P_2} = \frac{c_2\,(P_1 + S_1) - s_1\, S_1}{a_2\, A_2 - c_2\,(A_2 - P_2)}$$

Multiplying out gives

$$P_1\, P_2\, c_2 + P_2\, c_2\, S_1 - P_2\, s_1\, S_1 = P_1\, a_2\, A_2 - P_1\, c_2\, A_2 + P_1\, c_2\, P_2$$

$$\therefore \quad P_2\, c_2\, S_1 + P_1\, c_2\, A_2 = P_2\, s_1\, S_1 + P_1\, a_2\, A_2$$

$$\therefore \; c_2\,(P_2\, S_1 + P_1\, A_2) = s_1\,(P_2\, S_1) + a_2\,(P_1\, S_2).$$

Dividing throughout by P_1 and rearranging gives

$$a_2\, A_2 + s_1\left(\frac{P_2}{P_1} \times S_1\right) = c_2\left(A_2 + \frac{P_2}{P_1} \times S_1\right) \quad \ldots \quad \ldots \quad (15)$$

The similarity to equation (2) in section 1.2 will be obvious, and it will be seen that the compound pinions do no more than modify the teeth in the sun wheel by their own ratio. This so-called compound train can, therefore, be reduced to an equivalent basic train by the simple expedient of making $S_2 = \frac{P_2}{P_1} \times S_1$.

Here again the example has been worked in full to provide practice in the use of equations.

2.7 Conclusions.

There is no need to proceed further with the analysis of trains and the following conclusions may now be drawn.

(i) One equation suffices to define all the relative movements of the three coaxial members of the basic train (eq. 2).

(ii) For two or more coupled trains, all four equations may be required, depending on whether any of the members are missing.

(iii) For all coupled trains it is axiomatic that there must be two couplings for every added train in order to produce a differential train.

(iv) To convert such a differential train into a train giving an absolute ratio between two members a reaction member is required, and this reaction member may be one of the coupled pairs.

(v) The number of absolute ratios obtainable from a differential train of P coaxial members is given by:—

$$P \times \frac{Q!}{2 \times (Q-2)!} \text{ where } Q = P - 1.$$

This is the normal equation for a permutation of this type.

(vi) It should be borne in mind that in certain cases the numbers of teeth in the various members may be such that some of the ratios can be repeated. Extreme examples would be if Cases 1, 3, 11 and 13 in Chart II had identical trains. In each case a simple basic train of only three ratios would result. In a lesser degree this can occur in many of the other cases.

CHAPTER 3

SYNTHESIS OF TRAINS

3.1 The analysis of known epicyclic trains has been shown to be comparatively straightforward. This is so because the train already exists, the couplings are known, and input and output members are given and the numbers of teeth are either known or can be counted. When the procedure is reversed and a train is required to meet given conditions, certain difficulties arise.

Taking the basic train, it is usually possible by inspection to select gear proportions to give required ratios if these lie within the range of a single train. Assuming that $A = NS$, the three equations (5), (6) and (7) in 1.3 can be written down as:

S stationary: $aNS = c\,(NS + S)$

$$i.e., \frac{a}{c} = \frac{NS + S}{NS} = \frac{N + 1}{N} \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

C stationary: $aN = -s$

$$i.e., \frac{a}{s} = -\frac{1}{N} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

A stationary: $sS = c\,(NS + S)$

$$i.e., \frac{s}{c} = \frac{NS + S}{S} = N + 1 \ldots \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

It is not usual for N to be less than about 2 or more than about 9 so that a glance at these three equations will reveal whether any required ratio can be given by a basic train. For example, if a 4-to-1 reduction is required, with a stationary annulus, equation (18) shows $N = 3$, *i.e.*, $A = 3S$.

For ratios lying outside the range of $N = 2$ to $N = 9$, a coupled train is indicated. The selection of coupled trains is not always easy, but the method which follows will usually produce the best solution. (Refer again to " Compound Trains " 2.6 and to 3.3.)

3.2 Consider the general case of a gearbox having a casing Z from which project two coaxial shafts X and Y, X being the input member and Y the output member. The gearing connecting X and Y is

unknown and it can be of any type. Let the speeds of the three members be x, y, and z; and when $z = 0$, let $\frac{x}{y} = r$, this being any required ratio and either positive or negative in sign.

Rotate X through $+1$ revolution whilst holding Z stationary and then rotate the whole mechanism -1 revolution. The relative movements of the elements so obtained are tabulated and added.

(Revs of X)	(Revs of Y)	(Revs of Z)
$+1$	$+\frac{1}{r}$	0
-1	-1	-1
0	$\frac{1-r}{r}$	-1

Multiply the first line by x and the sum by $-z$ so as to bring in relative speeds and again add

(Revs of X)	(Revs of Y)	(Revs of Z)
$+x$	$+\frac{x}{r}$	$+0$
0	$-\frac{z(1-r)}{r}$	$+z$
$+x$	$\frac{x - z(1-r)}{r}$	$+z$

By definition this gives, when x and z are the speeds of X and Z,

$$y = \frac{x - z(1-r)}{r}$$

i.e., $z(1-r) + yr - x = 0$ ... ... (19)

This is the general characteristic equation for an unknown epicyclic train. Reverting to equation (2) this can be transformed from

$$aA + sS = c(A + S)$$

into $a\left(\frac{A}{A+S}\right) + s\left(\frac{S}{A+S}\right) - c = 0$ ... ... (19a)

Putting $\frac{S}{A+S} = \mathrm{R}$ we obtain,

$a(1 - \mathrm{R}) + s\mathrm{R} - c = 0$ ... ... (20)

The connection between equations (19) and (20) is obvious. In equation (20) the geometry of the train shows that $R < \frac{1}{2}$ therefore $1 > (1 - R) > R$. The unity coefficient governs the cage; R, the smallest coefficient, governs the sun; and the remaining coefficient governs the annulus. To bring the indeterminate equation (19) into line with (20) it is only necessary to insert any required value of r and then divide throughout by the largest resulting coefficient. Using the example in 3.1, $r = 4$. Inserting this in (19) gives

$$z(-3) + y(4) - x = 0 \qquad \text{or} \qquad -\frac{3}{4}z + y - \frac{1}{4}x = 0$$

From the above reasoning, this identifies x as the sun and input member, y as the carrier and output member, and z as the annulus and reaction member. Also

$$R = \frac{1}{4} = \frac{S}{A + S}. \quad \text{Hence, } 4S = A + S \text{ and } A = 3S \text{ as before.}$$

3.3 Trains lying outside the range of the basic train can now be considered. Such trains may be outside the limits already stipulated in 3.1, or space reasons may make these limits even closer. Further, it may be stipulated that the smallest permissible pinion must not have less than so many teeth. In fact, the variations possible are so great that each problem must normally be considered on its own merits, and it is then that full use of equations (1) to (4), (19) and (20) must be made, with attention paid to the observations made on compound trains in section 2.6.

3.4 A fully worked example for a gearbox requiring coupled trains will show the method of utilizing the equations. A gearbox is required to the following specification. The casing must be fixed, with input and output shafts at opposite ends and the shafts must be coaxial. Two ratios are required, a 4-to-1 reduction in the same sense as the input and a 4-to-1 reduction in the opposite sense to the input. The gears must be brought into operation by frictional restraints so that when the input shaft is rotating, either of the ratios can be brought into operation from rest. (Such a gearbox could be used with suitable control gear as a tapping head for either right-hand or left-hand threads.)

Here we have $r = +4$ or -4. The substitution of these values in equation (19) yield, for $r = +4$

$$-3z + 4y - x = 0 \quad \text{or} \quad -\frac{3}{4}z + y - \frac{1}{4}x = 0 \ldots \qquad \ldots \qquad \text{(i)}$$

Using the method of identification explained in section 3.20 shows the annulus to be the reaction member, the cage the output member, and the sun the input member. $R = \frac{1}{4}$, whence $A = 3S$.

For $r = -4$

$$+ 5z - 4y - x = 0$$

or, $+ z - \frac{4}{5}y - \frac{1}{5}x = 0$ ... ... (ii)

Identifying as in section 3.2 shows the cage to be the reaction member, the annulus the output member and the sun the input member. $R = \frac{1}{5}$, whence $A = 4S$.

Both of these trains are within the range of a basic train and either could be used. Since the annulus is usually the easiest member to restrain by frictional means, train (i) would normally be chosen. In order to obtain the ratio $r = -4$ while keeping the same input and output members it will be obvious that some motion must be imparted to the annulus so as to give the relationship $s_1 = -4c_1$ or $c_1 = -\frac{s_1}{4}$.

Since the characteristic equation (2) of the basic train in this case can be written as $a_1 \times 3S_1 + s_1 S_1 = c_1 (3S_1 + S_1)$, this gives $3a_1 + s_1 = 4c_1$. The two trains can now be considered together.

(i) a. When $s_1 = -4c_1$

$$3a_1 - 4c_1 = 4c_1$$

i.e. $3a_1 = 8c_1$

and $\frac{a_1}{c_1} = \frac{8}{3} = r$.

Substitute this value for r in equation (19)

$$-\frac{5}{3}z + \frac{8}{3}y - x = 0.$$

i.e. $-\frac{5}{8}z + y - \frac{3}{8}x = 0.$

The sun is x, the cage y and the reaction member is the annulus. $R = \frac{3}{8}$ whence $A_2 = \frac{5}{3} \times S_2$.

$$\frac{x}{y} = \frac{s_2}{c_2} = \frac{8}{3} = \frac{a_1}{c_1}$$

S_2 is coupled to A_1, C_2 to C_1, A_2 is the reaction member, S_1 is the input and C_1 the output. The train is arranged as shown on the left in Fig 3.

(i) b. When $c_1 = \frac{s_1}{4}$

$$3a_1 + s_1 = -s_1$$

i.e. $3a_1 = -2s_1$

and $\frac{a_1}{s_1} = -\frac{2}{3} = r$.

Substitute this value for r in equation (19).

$$\frac{5}{3}z - \frac{2}{3}y - x = 0$$

i.e. $z - \frac{2}{5}y - \frac{3}{5}x = 0.$

The sun is y, the reaction member is the cage and the annulus is x. $R = \frac{2}{5}$ whence $A_2 = \frac{3}{2}S_2$.

$$\frac{x}{y} = \frac{a_2}{s_2} = -\frac{2}{3} = \frac{a_1}{s_1}$$

A_2 is coupled to A_1, S_2 to S_1, C_2 is the reaction member, S_1 is the input and C_1 the output. The train is arranged as shown on the right in Fig 3.

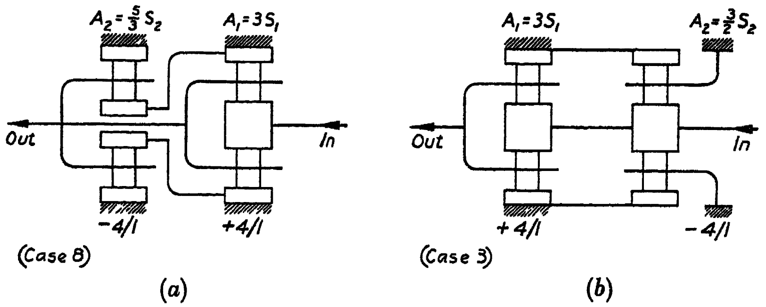


Fig 3. Arrangements of thc Two Solutions for $s_1 = -4e_1$ and for $e_1 = s_1/4$

Had we accepted train (ii) a parallel reasoning would have revealed two further trains, giving the same result. These are shown in Figs 4 (*a*) and 4 (*b*).

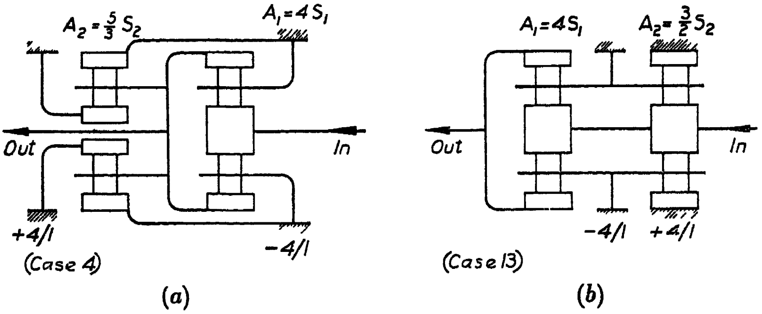


Fig 4. Arrangements for Two Further Solutions

We have thus arrived at no less than four gear trains, each consisting of two double-coupled basic trains, all four of which comply with the conditions first laid down. Selection of the best train would next be made on the grounds of simplicity, ease of manufacture and so on. The trains shown at the right in Fig. 3 (b) and in Fig 4 (*b*) would appear to be the most straightforward even though the ratio $A_2 = 1.5\ S_2$ is rather low.

3.5 Ratios when R Approaches 0·5

When $x/y = r = \frac{1}{2}$, 2 or -1, or approaches any of these values, substitution in equation (19) leads to an impossible basic train or

at any rate one badly out of balance. Put $r = \frac{1}{2}$ and the result is $\frac{1}{2} z + \frac{1}{2} y - x = 0$. Similarly $r = 2$ gives $-\frac{1}{2} z + y - \frac{1}{2} x = 0$ and $r = -1$ gives $z - \frac{1}{2} y - x = 0$. In each case only the cage can be identified and since R $= 1 -$ R neither the annulus nor sun can be identified. When R approaches one of these values, R can be identified but usually S so nearly equals A that the planet pinions are too small. Three courses are open for investigation when this occurs and these will be dealt with in turn.

(a) Where the required ratio is one of several, an existing train can be coupled to an additional train by the methods given in section 3.4.

(b) One obvious solution is to make a layshaft type of gear as shown in Fig 5. This is in effect a " compound " train less both annuli, the cage being the reaction member. Tooth proportions are $r = S_1/P_1$ and $P_2 = S_2 = \frac{P_1 + S_1}{2}$. Where four differing pinions give more scope for choosing a precise ratio, tooth proportions can be made such that $r = S_1 P_2/P_1 S_2$ and $S_1 + P_1 = S_2 + P_2$. This train does not give reverse ratios.

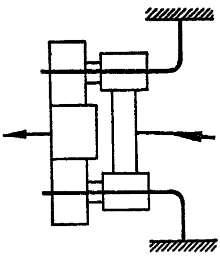


Fig 5. Layshaft Type of Gear

(c) A not-so-obvious solution can again be found from Case 11 of Chart II.

In section 2.6 and by equation (15) it was shown that for this type of train $a_2 A_2 + s_1 (P_2/P_1 \times S_1) - c_2 (A_2 + P_2/P_1 \times S_1) = 0$. From this R $= \frac{P_2/P_1 \times S_1}{A_2 + P_2/P_1 \times S_1}$ since, in accordance with the expedient indicated in section 2.6, $S_2 = \frac{P_2 S_1}{P_1}$. Now make $A_2 = 2S_1$ and $P_2/P_1 = n$. This gives R $= \frac{n S_1}{2S_1 + nS_1} = \frac{n}{2 + n}$ whence $n = \frac{2\text{R}}{1 - \text{R}}$. Since $A_2 = P_2 + P_1 + S_1$, and P_1 is obviously the

smallest pinion, we can write the train proportions as

$$P_2 = \frac{2R}{1-R} \times P_1 \; ; \; S_1 = \frac{1+R}{1-R} \times P_1 \; ; \text{ and } A_2 = \frac{2(1+R)}{1-R} \times P_1.$$

For ease of factorization it is sometimes convenient to make $A_2 = NS_1$ where N is of any value approximating to 2, so making $n = \frac{NR}{1-R}$. For example, if $N = \frac{1}{R}$, n becomes $\frac{1}{1-R}$.

For a ratio $r = \frac{20}{9}$ we get $A = (11/9)S$ for a basic train. Assuming the smallest pinion to be 11 teeth this gives $S = 99$ and $A = 121$. Using the "compound" train as detailed above we get $P_1 = 11$, $P_2 = 18$, $S_1 = 29$ and $A_2 = 58$, in a much more compact train.

3.6 Idler Pinions

The use of idler pinions in epicyclic gearing produces rather unexpected results. Taking the basic train and substituting a meshing pair of planets for the single planet pinions as shown in Fig 6 we actually have a pair of trains double-coupled as shown in Case 12 Chart II, with the sun of one train and the annulus of the other missing. The characteristic equations for this train can be obtained either from the basic train equations by making $c_1 = c_2$ and $p_1 = p_2 \times P_2/P_1$ or from first principles. Either method will produce the relationship:

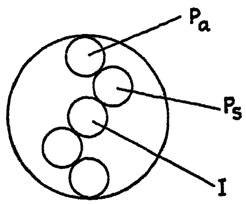


Fig 6. Arrangement of Epicyclic Train with Idler Pinions

$$aA - sS = c(A - S) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

This should be compared with equation (2). Also, by putting $A = NS$ in this type of train and comparing it with the basic train we get

Reaction	Ratio	Basic	Idler
Sun	a/c	$+\frac{N+1}{N}$	$+\frac{N-1}{N}$
Cage	a/s	$-\frac{1}{N}$	$+\frac{1}{N}$
Annulus	c/s	$+\frac{1}{N+1}$	$-\frac{1}{N-1}$

Also, for equation (19a) we can write

$$a\left(\frac{A}{A-S}\right) - s\left(\frac{S}{A-S}\right) - c = 0 \quad \dots \quad \dots \quad \dots \quad (21)$$

$\frac{A}{A-S}$ is greater than unity and $\frac{S}{A-S}$ less than unity so that to bring the equation into line with equation (20) we can rearrange to

$$a - s \times \frac{S}{A} - c\,\frac{A-S}{A} = 0 \quad \dots \quad \dots \quad \dots \quad \dots \quad (22)$$

The unity coefficient governs the annulus, the smallest the sun, and the intermediate the cage. R now equals S/A provided $A > 2S$. When $A = 2S$ we get $\frac{S}{A} = \frac{A-S}{A}$ and here we have an alternative solution for ratios where R $= 1/2$ or nearly so, since equation (22) can be used.

A train for $r = \frac{20}{9}$ alternative to that given in section 3.5 would be

$S = 27$, P_s and $P_a = 11$ and $A = 60$.

3.7 High Reduction Gears

Another example of the versatility of Case 11 trains is the provision of high reduction ratios. The efficiency is often low, however, and in many cases the drive is not reversible to give high-increase ratios. Consideration of this type of train will show that when the two sun wheels or the two annuli are missing and the cage is used as the input member, the ratio c/s or c/a can be very large. Also, depending on which of the two suns or two annuli are used as the reaction member the output can be in the same sense or in the opposite sense to the input. The characteristic equations to the two trains for the gear shown in Fig 7 are:

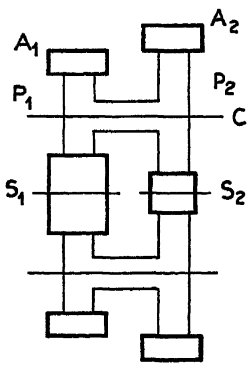


Fig 7. A High Reduction Gear

Less Sun Wheels

$$a_1 A_1 - a_2\left\{A_2 \times \frac{P_1}{P_2}\right\} \qquad - c\left\{A_1 - A_2 \times \frac{P_1}{P_2}\right\} = 0 \quad (23)$$

$$\text{and } a_2 A_2 - a_1\left\{A_1 \times \frac{P_2}{P_1}\right\} \qquad - c\left\{A_2 - A_1 \times \frac{P_2}{P_1}\right\} = 0 \quad (24)$$

Less Annuli

$$s_1 S_1 - s_2\left\{S_2 \times \frac{P_1}{P_2}\right\} \qquad - c\left\{S_1 - S_2 \times \frac{P_1}{P_2}\right\} = 0 \quad (25)$$

$$\text{and } s_2 S_2 - s_1\left\{S_1 \times \frac{P_2}{P_1}\right\} \qquad - c\left\{S_2 - S_1 \times \frac{P_2}{P_1}\right\} = 0 \quad (26)$$

From equation (23) if A_1 is the reaction member we have

$$a_2\left\{A_2 \times \frac{P_1}{P_2}\right\} = c\left\{A_1 - A_2 \times \frac{P_1}{P_2}\right\}$$

$$\textit{i.e.}\ \frac{c}{a_2} = \frac{(A_2\, P_1)/P_2}{A_1 - (A_2\, P_1)/P_2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (27)$$

Let A_1 approach A_2 and let the tooth numbers be 95 and 100 respectively. From the geometry of the train, $A_2 - A_1 = P_2 - P_1$ so that convenient teeth for P_1 and P_2 could be 35 and 40. The ratio then becomes

$$\frac{c}{a_2} = \frac{100 \times \dfrac{35}{40}}{95 - \dfrac{100 \times 35}{40}} = 11{\cdot}33$$

If A_2 were made the reaction member it will be found that the ratio c/a_1 becomes $-12{\cdot}6\dot{6}$.

The extreme case of this class of gear is when there is only one tooth difference in the two annuli, one of them being cut " one tooth less " in a standard diameter ring. The planet pinion is a single pinion engaging both annuli. Equation 5 then reduces to:

$$\frac{c}{a_2} = \frac{A_2}{A_1 - A_2}.$$

Using teeth of $A_1 = 99$ and $A_2 = 100$ we get a reduction of $-100/1$, or with A_2 as the reaction member a reduction of $+99/1$. Such a gear would not be reversible but it is of use in counting mechanisms and the like.

3.8 Tooth Numbers for Assembly

The methods described for the synthesis of the various trains only give the correct *proportions* of the various parts. In allocating teeth to suit these proportions certain other factors must be borne in mind. Taking the basic train with A, P and S teeth and only one planet then $P = \dfrac{A - S}{2}$ and $A - S$ must be an even number for P to be a standard pinion. If $A - S$ is an odd number then it is usual to cut the next lowest number of teeth in the blank size found as above. Since there is usually more than one planet, consider the general case with n planets. In this case, with the planets equally spaced, the orientation of the planet teeth which mesh with the sun is fixed and it follows that the orientation of the teeth which mesh with the annulus must also be fixed and must agree with the spacing of the annulus teeth. The obvious solution is to make both A and S divisible by n and this is the method usually adopted. If this is not possible a further solution is as follows.

With the sun fixed rotate the carrier by $1/n$th revolution. Since $aA = c(A + S)$ we have $a = c\dfrac{(A + S)}{A} = \dfrac{A + S}{nA}$. In moving the carrier $1/n$th revolution the planets advance one step and must again be in identical phasing. It follows that the gear ring must advance an exact number of teeth so that a can be given the value $1/A \times$ an integer.

$$\text{hence,} \quad \frac{A + S}{nA} = \frac{1}{A} \times \text{an integer;}$$

$$\text{and,} \quad \frac{A + S}{n} = \text{an integer.}$$

Thus, provided $A + S$ is evenly divisible by n, the train will assemble even if A and S considered alone are not so divisible.

Where compound planet pinions are used it is essential that a pair of teeth on the planets are in absolute alignment and, although a similar process of reasoning to that given for a basic train allows two solutions, it is preferable that the tooth numbers for both sun and annulus are divisible by n. For assembly, the aligning teeth on the planets should be marked and assembled either radially outwards from the sun or radially inwards from the annulus.

3.9 Alternative Couplings

In all the foregoing examples two assumptions have been made, these being that the gears are obtained by applying frictional restraints to the reaction members and that the coupling between the trains is unaltered. Although epicyclic gears lend themselves to this treatment, neither of these conditions is essential. For example, using a constant output member, the input can be taken to alternative members by means of dog clutches, and *vice versa.* Or, keeping the input and output members the same, the inter-coupling between the trains can be varied, once again by alternative clutching. Further, both these variations can be employed together to provide additional ratios to those given by the main coupled trains. Alternatively, the coupling between two trains can be disconnected and one of the parts thus freed can become a reaction member so as to leave only one train in use. The number of such variations is without limit and it would be quite impossible to tabulate them. Much ingenuity has been exercised in the past in evolving combinations to suit given requirements and the number of patents taken out for them must run into many hundreds, if not thousands.

3.10 Conclusions

The following conclusions can now be drawn on the synthesis of epicyclic trains to give specified ratios.

(a) For ratios within the range of a basic train the equations in 3.1 will give train proportions by inspection.

(b) For ratios outside the range of a basic train coupled trains must be used, selection being made by the methods given in 3.4.

(c) For ratios where the method in 3.4 leads to difficult or impossible trains, recourse must be made to the methods given in 3.5, 3.6 or 3.7.

(d) Due to the infinite variety of possible trains it is impossible to lay down hard and fast equations for all combinations. Applying the methods given will usually lead to several solutions, the final choice being made on the score of simplicity, cost and so on.

CHAPTER 4

THE WILSON TYPE GEARBOX

To show that all the necessary information for the synthesis of a series of epicyclic gear trains has been given in the foregoing chapters, the rather elaborate example of the well-known Wilson gearbox will be given. This example was deliberately chosen to show that, despite its apparent complication, when the problem is expounded it is really easy to solve. The problem confronting the designer was to produce a gearbox with four forward gears, one of them being direct, and a reverse gear. The input and output shafts are of necessity coaxial and at opposite ends of the box. The gears are to be selected by frictional means without dog clutches. The ratios rather depend upon the engine and car, but for simplicity of figuring the following even ratios will be chosen in this example:—

Low Gear: $r_1 = 4/1$
2nd Gear : $r_2 = 2/1$
3rd Gear : $r_3 = 1{\cdot}5/1$
Top Gear : $r_4 = 1/1$
Reverse : $r_5 = -4/1$

(These ratios are of the same general magnitude as the actual gearbox ratios.)

Any other values could, of course, be chosen without affecting the method. It is not possible to use symbols and state a general case since the train proportions cannot be visualized.

4.1 General Equations and Low Gear

Applying equations (19) and (20) and substituting the above values of r in (19) we get:—

Low. $z\,(1 - 4) + 4y - x = 0$ ... ... (28)

hence $-\,0{\cdot}75z + y - 0{\cdot}25x = 0$

x is the sun, y the cage and z the annulus. $R_1 = 0{\cdot}25$, giving $A_1 = 3S_1$.

2nd Gear. $z\,(1 - 2) + 2y - x = 0$ ... ... (29)

hence $0{\cdot}5z + y - 0{\cdot}5x = 0$.

This requires a coupled train as explained in 3.5.

3rd Gear. $z(1 - 1{\cdot}5) + 1{\cdot}5y - x = 0$ ... ... (30)

hence $0{\cdot}333z + y - 0{\cdot}666x = 0$.

z is the sun, y the cage and x the annulus. $R_3 = {\cdot}333$, giving $A_3 = 2S_3$.

Top. $z(1 - 1) + y - x = 0$ ... ... (31)

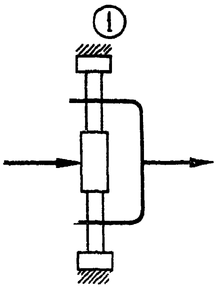
$$y = x$$

This signifies that either the input shaft is connected direct to the output shaft or that the entire gear is locked so as to rotate *en bloc.*

Reverse. $z\left\{1 - (-4)\right\} - 4y - x = 0$ ... ... (32)

i.e. $z - 0{\cdot}8y - 0{\cdot}2x = 0$.

x is the sun, z the cage and y the annulus. $R_5 = 0{\cdot}2$, giving $A_5 = 4S_5$.



$A_1 = 3S_1$

Fig 8. Basic Train for "Low" Gear

This gives us three equations all within the range of a basic train. Since in a gearbox of the type visualized the annulus is the easiest member to which a band-brake can be applied, the obvious choice for the first basic train is "low" gear. This is shown diagrammatically in Fig 8.

4.2 Second Gear

Now, to vary the ratio of input to output we must release the annulus and then drive it at some predetermined speed relative to either the input or output shafts. The characteristic equation for this train is given by equation (2) which can be written as:—

$$a_1\, 3S_1 + s_1\, S_1 = c_1\,(3S_1 + S_1)$$

i.e. $3a_1 + s_1 = 4c_1$ ... ... (33)

The required value of $\frac{s_1}{c_1} = 2$ so that $s_1 = 2c_1$ or $c_1 = \frac{s_1}{2}$.

Inserting these values in (33) we get

$3a_1 + 2c_1 = 4c_1$	or, $3a_1 + s_1 = 2s_1$
i.e. $\frac{c_1}{a_1} = \frac{3}{2}$ ... ... (34)	*i.e.* $\frac{s_1}{a_1} = 3$ ... ... (35)

If we drive the annulus from the cage, equation (34) applies; and if from the sun, equation (35) applies. We thus require a train whose ratio $x/y = r_2 = 3/2$ or 3. Substituting these values in equation (19) we get

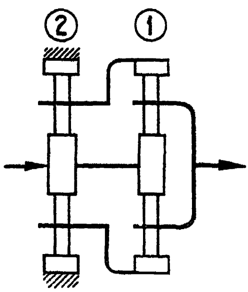
$$z\,(1 - 3/2) + 3/2y - x = 0$$
$$i.e.\ -1/3z + y - 2/3x = 0.$$

y is the cage, x the annulus and z is the sun. R $= 1/3$, giving $A_2 = 2S_2$.

or, $$z\,(1 - 3) + 3y - x = 0.$$
$$i.e.\ -2/3z + y - 1/3x = 0.$$

y is the cage, z the annulus and x the sun. R $= 1/3$, giving $A_2 = 2S_2$.

Either of these trains is within the range of a simple train, but for the reason already given, that in which the annulus is the reaction member would be chosen. It is entirely fortuitous that the trains are of the same proportions. Had the value of s_1/c_1 been other than 2 they would have been dissimilar. Accepting this train, the coupling to the train in Fig 9 is such that the two suns are coupled and the cage of the second train is coupled to the annulus of the first train. This can be identified as a Case 7 train from Chart II of Chapter 1.



$A_2 = 2S_2$

Fig 9. Coupling for First and Second Gears

4.3 In seeking 3rd gear we have two alternatives. We can add a train alternative to the one required for 2nd gear and drive the first gear annulus, or we can add a train in addition to it and drive the annulus of the 2nd gear train. Taking the first alternative since it reduces the number of trains in use at one time, we must again drive the annulus of the first train at a speed which will give $s_1/c_1 = 1{\cdot}5$. This gives $s_1 = 1{\cdot}5c_1$ or $c_1 = s_1/1{\cdot}5$. Inserting these in equation (33) gives:—

$$3a_1 + 1{\cdot}5c_1 = 4c_1$$
$$\text{or } \frac{c_1}{a_1} = 1{\cdot}2.$$

or, $$3a_1 + s_1 = 2{\cdot}66s_1$$
$$i.e.\ \frac{s_1}{a_1} = 1{\cdot}8.$$

We thus require a train whose ratio $x/y = r_3 = 1{\cdot}2$ or $1{\cdot}8$ and by substituting these in equation (19) we obtain

$$z\,(1 - 1{\cdot}2) + 1{\cdot}2y - x = 0$$
$$i.e.\ -1/6z + y - 5/6x = 0.$$

Here y is the cage, x the annulus and z is the sun. $R = 1/6$, giving $A_3 = 5S_3$.

or, $$z\,(1 - 1{\cdot}8) + 1{\cdot}8y - x = 0.$$
$$i.e.\ -4/9z + y - 5/9x = 0.$$

Here y is the cage, x the annulus and z is the sun. $R = 4/9$, giving $A_3 = 1{\cdot}25S_3$.

The ratio $A_3/S_3 = 1{\cdot}25$ is too small so that the train with $A_3 = 5S_3$ is the choice. The drive is in to the annulus and out from the cage with the sun as reaction; cage 3 is coupled to cage 2 and the annulus 3 is coupled to the input shaft. The train could be arranged as in Fig 10. The couplings are rather awkward and it would appear that investigation of the driving of the second gear annulus is worthwhile.

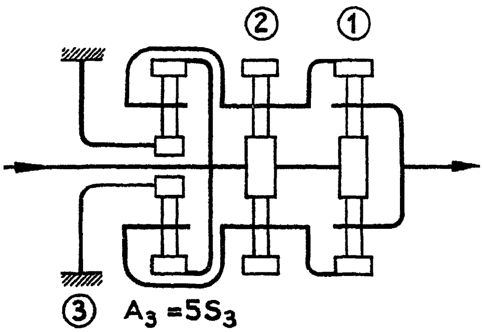


Fig 10. Coupling for First, Second and Third Gears

4.4 Third Gear Alternatives

We must now drive A_2 at such a speed as will give $s_1/c_1 = 1.5$ as before, and since we can only drive it from s_2 or c_2 we require the ratios of s_2/a_2 and c_2/a_2, which will give $s_1/c_1 = 1{\cdot}5$. Further, the couplings in Fig 9 show that $s_1 = s_2$ and $a_1 = c_2$.

$$3a_1 + s_1 = 4c_1 \qquad \text{(Equation 33)}$$

that is $3c_2 + s_2 = \dfrac{4 \times s_2}{1{\cdot}5}$

or $c_2 = 5/9s_2$ ... ... (36) or $s_2 = 9/5c_2$ ... ... (37)

Now the characteristic equation of train 2 is

$$2a_2 + s_2 = 3c_2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (38)$$

and using the values in (36) and (37) we can write

$2a_2 + s_2 = 5/3s_2$ or $2a_2 + 9/5c_2 = 3c_2$

i.e. $\dfrac{s_2}{a_2} = 3$ *i.e.* $\dfrac{c_2}{a_2} = 1{\cdot}66$

We thus require a train whose ratio $x/y = r = 3$ or $1{\cdot}66$ and by substituting these in equation (19) as before we get

$z\,(1 - 3) + 3y - x = 0$ or $z\,(1 - 1{\cdot}66) + 1.66y - x = 0$

i.e. $-2/3z + y - 1/3x = 0$; where y is the cage, x the sun and z the annulus. R $= 1/3$, giving $A = 2S$.	*i.e.* $-2/5z + y - 3/5x = 0$; with y as the cage, z the sun and x the annulus. R $= 2/5$, giving $A = 1{\cdot}5S$.

Both of these trains are within the range of a basic train and the couplings are as shown in Fig 11 (*a*) and (*b*).

The designer of the Wilson box chose Fig 11 (*b*), presumably because his ratio was not 1·5/1, although Fig 11 (*a*) gives an annulus as the reaction member.

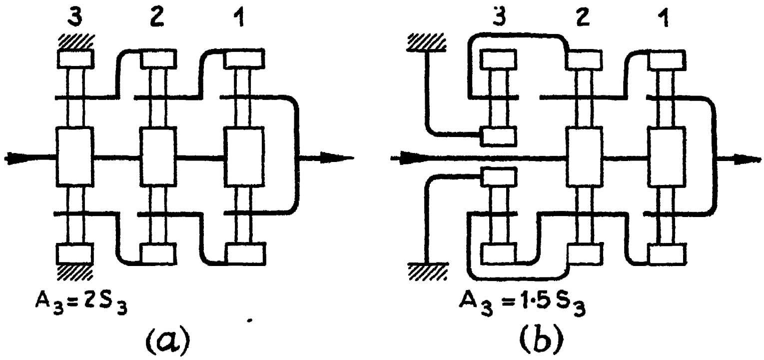


Fig 11. Third Gear Alternatives (*a*) for $A_3 = 2S_3$; (*b*) for $A_3 = 1{\cdot}5S_3$.

4.5 Reverse

Since high gear entails the locking of gears to produce a solid drive, reverse will be dealt with next so that the complete gear trains will be present. Inspection of any of the trains shown in Figs 10 or 11 show that the easiest train to modify is the low gear train, and the annulus A_1 must therefore be driven at such a speed as to make $s_1/c_1 = -4$, *i.e.* $s_1 = -4c_1$ or $c_1 = -\frac{s_1}{4}$.

From (33), $3a_1 + s_1 = 4c_1$ we have

$3a_1 - 4c_1 = 4c_1$ or $3a_1 + s_1 = -s_1$

i.e. $\frac{c_1}{a_1} = \frac{3}{8}$ *i.e.* $\frac{s_1}{a_1} = -\frac{3}{2}$

We thus require a train whose ratio $\frac{x}{y} = r = \frac{3}{8}$ or $-\frac{3}{2}$.

Substituting these values in equation (19) as before we get

$z\,(1 - 3/8) + 3/8y - x = 0$ or $z\,[1 - (-3/2)] - 3/2y - x = 0$

i.e. $5/8z + 3/8y - x = 0$; where x is the cage, z the annulus and y the sun. $R = 3/8$, giving $A_4 = 1.66S_4$.

i.e. $z - 3/5y - 2/5x = 0$; where z is the cage, y the annulus and x the sun. $R = 2/5$, giving $A = 1{\cdot}5S$.

As before, the train in Fig 12 (*a*) would be chosen since the annulus is the reaction member.

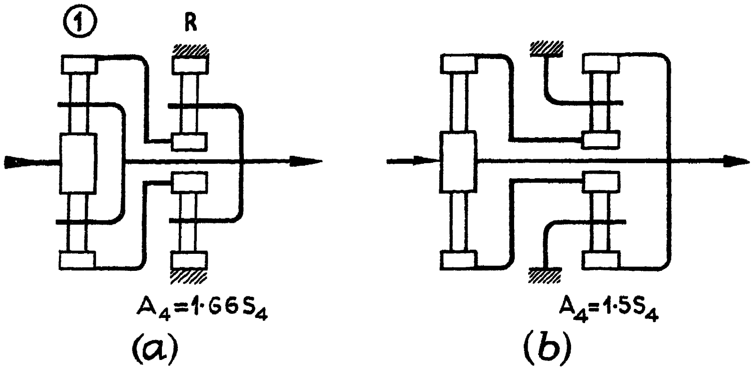


Fig 12. Reverse Gear Arrangements (*a*) for $A_4 = 1{\cdot}66S_4$; (*b*) for $A_4 = 1{\cdot}5\ S_4$.

4.6 Direct Drive

It only remains now to decide on how to lock the gears to produce direct drive. Referring to Fig 11 (*b*), which shows the nearly complete gear, it will be seen that the easiest place to do this is to put a clutch between the input shaft and the inner side of the 3rd gear sun reaction drum. Doing this and adding in Fig 12 (*a*) gives the complete gearbox as shown in Fig 13.

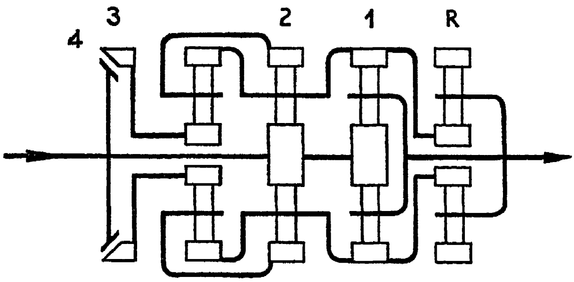


Fig 13. Method of Locking the Gears to Produce the Direct Drive.

4.7 Analysis Diagrams

It is convenient here to show a diagrammatic method of charting a complex gear train for analysis. Fig 14 shows the Wilson gearbox of Fig 13 in this form. The four trains are shown as the symbols *A*, *C*, *P* and *S*. The ringed members are alternative reaction

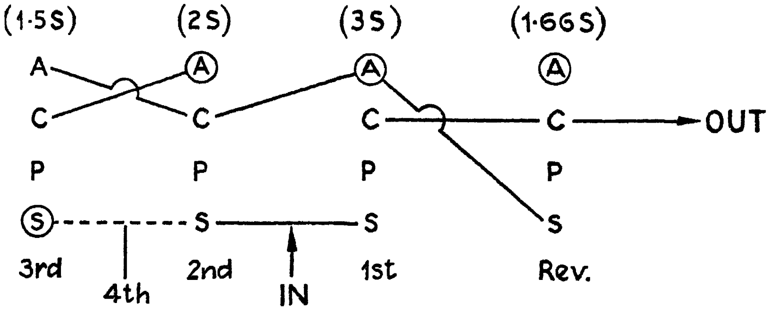


Fig 14. Analysis Diagram for Complete Wilson Gear

members, the full lines show permanent couplings and dotted lines show detachable couplings. The input and output points are marked and the brackets above the A symbols give the train proportions. Using this chart with the analysis equations the sequence of analysis becomes easy. For instance, if it is required that the ratio of second gear be calculated, a glance at Fig 14 enables that train to be abstracted as shown in Fig 15. This diagram should be compared with Fig 9.

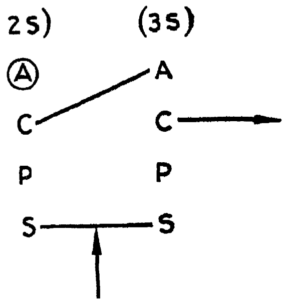


Fig 15. Abstract of the Complete Diagram shown in Fig 14 to enable the Ratio of the Second Gear to be calculated

4.8 Rapid Analysis

2nd Gear.—Equation (2) gives us the general equation of any basic train

$aA + sS = c\ (A + S)$

Since for train one, $A_1 = 3S_1$ we can write

$$3a_1 + s_1 = 4c_1 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (39)$$

and for train two, where $A_2 = 2S_2$, we can write

$$2a_2 + s_2 = 3c_2 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (40)$$

Since $c_2 = a_1$, $s_2 = s_1$ and $a_2 = 0$ this gives

$0 + s_1 = 3a_1$

or $a_1 = s_1/3$.

Substituting this in (39) we have

$$s_1 + s_1 = 4c_1$$
$$\therefore\ 2s_1 = 4c_1$$
$$\therefore\ s_1/c_1 = 2.$$

3rd Gear.—Similarly for 3rd gear the three characteristic equations are written down as

(3rd) $1{\cdot}5\, a_3 + 0 = 2{\cdot}5c_3$ ... ... (41)

(2nd) $2a_2 + s_2 = 3c_2$ ... ... (42)

(1st) $3a_1 + s_1 = 4c_1$ ... ... (43)

From (41) $1{\cdot}5a_3 = 2.5c_3$; hence $1{\cdot}5c_2 = 2{\cdot}5a_2$. Using this relation to eliminate a_2 from (42) gives

$$\frac{3}{2{\cdot}5}c_2 + s_2 = 3c_2;\ \text{or } s_2 = \frac{9}{5}c_2.$$

Using this to eliminate a_1 from (43) gives

$$\frac{5}{3}\,s_1 + s_1 = 4c_1;\ \text{or } \frac{8}{3}\,s_1 = 4c_1 \text{ so giving the ratio } s_1/c_1 = 1{\cdot}5.$$

This same procedure can be applied for reverse gear and indeed to any series of coupled trains. It is essential, however, that the derivation of the velocity equations given in Chapters 1 and 2 is fully understood before facility in the manipulation of the further characteristic equations can be acquired.

CHAPTER 5

GENERAL EQUATIONS

5.1 Train Proportions and Ranges

It was stated at the beginning of Chapter 4 that it was not possible to state a general case when synthesising a train because the train proportions and couplings could not be visualized. This is, of course, true, but after having arrived at a train it is often useful to evolve its general equation so that its ratio r can be given in terms of N. This then allows the range of r to be established between any given limits on N.

For instance, taking the second gear train of the Wilson box as it is used in Fig 9 we can write the characteristic of the two trains as:—

$$N_2\, a_2 + s_2 = (N_2 + 1)\, c_2 \quad \ldots \quad (44)$$

$$N_1\, a_1 + s_1 = (N_1 + 1)\, c_1 \quad \ldots \quad (45)$$

We know that $a_2 = 0$, $c_2 = a_1$ and $s_2 = s_1$, and the value of s_1/c_1 can be expressed in terms of N_1 and N_2 thus:—

From (44) $\quad s_1 = c_2\,(N_2 + 1)$;

$$\text{hence } a_1 = \frac{s_1}{N_2 + 1}$$

Substitute this value of a_1 in (45) giving

$\dfrac{N_1\, s_1}{N_2 + 1} + s_1 = c_1\,(N_1 + 1)$, which may be written as

$s_1\left\{\dfrac{N_1 + N_2 + 1}{N_2 + 1}\right\} = c_1\,(N_1 + 1)$, leading to

$$r_2 = \frac{s_1}{c_2} = \frac{(N_1 + 1)\,(N_2 + 1)}{N_1 + N_2 + 1} \quad \ldots \quad (46)$$

A similar process of reasoning applied to the third gear train will give the general equation for third gear as:—

$$r_3 = \frac{s_1}{c_1} = \frac{(N_1 + 1)\,(N_2 + 1)\,(N_3 + 1) - (N_1 + 1)\,N_2\,N_3}{(N_3 + 1)\,(N_1 + N_2 + 1) - N_2\,N_3} \quad (47)$$

and this shows the impossibility of attempting to arrive at a train in general terms. It is much better to say that ratios of *about* r_1, r_2 and so on are required, giving numerical values to r_1, r_2, etc.

Having arrived at a train or trains the ranges of r_1 and so on can be studied in relation to the appropriate values of N. Reverting to eq. (46), we require r_2 to equal 2 and N_1 already has the value of 3 so that:—

$$2\ (N_1 + N_2 + 1) = (N_1 + 1)\ (N_2 + 1)$$
$$\therefore 2\ (N_2 + 4) = 4\ (N_2 + 1)$$
$$\therefore 2N_2 + 8 = 4N_2 + 4$$

so that $N_2 = 2$, as before.

A more general form, knowing $N_1 = 3$, is:—

$$r_2 = \frac{4N_2 + 4}{N_2 + 4} \text{ or } N_2 = \frac{4r_2 - 4}{4 - r_2}$$

and from these two equations r_2 or N_2 values can be derived over any range of r_2 or N_2.

Similarly, knowing $N_1 = 3$ and $N_2 = 2$, equation (47) can be given the general form

$r_3 = \dfrac{4N_3 + 12}{4N_3 + 6}$, from which N_3 can be found for any value of r_3.

5.2 Characteristic Equations for Trains

It is also useful at times to have the complete characteristic equation for a given train showing the velocities of all the coaxial members without reference to input, output or reaction members, and to express in the same equation the train proportions in terms of N. It has already been shown that the equation for the basic train is $aN + s = c\ (N + 1)$. Again taking the Wilson second gear we can write

$$a_1\ N_1 + s_1 = c_1\ (N_1 + 1) \quad \ldots \quad (49)$$

and $a_2\ N_2 + s_2 = c_2\ (N_2 + 1)$ ... (50)

The couplings show that $c_2 = a_1$ and $s_2 = s_1$ and substituting in (50) gives

$$a_2\ N_2 + s_1 = a_1\ (N_1 + 1) \quad \ldots \quad (51)$$

Combining (51) and (49) and rearranging gives

$$a_2\ N_2 + 2s_1 = a_1(N_2 - N_2 + 1) + c_1\ (N_1 + 1) \quad \ldots \quad (52)$$

and this is the complete equation of the train. Here again if we make $a_2 = 0$, $s_1 = 2c_1$ and $N_1 = 3$ we can write

$$4c_1 = a_1\ (N_2 - N_1 + 1) + 4c_1$$

or $a_1\ (N_2 - N_1 + 1) = 0$ ... (53)

Since a_1 must have some value

$$N_2 - N_1 + 1 = 0$$

or $N_2 = N_1 - 1 = 2$ as before.

The complete characteristic equation for any series of basic trains can be obtained in this manner, but in the higher order trains they become very cumbersome. These equations are very useful, however, in assessing the possibilities of 2nd order trains as shown in Chart II, Chapter 2.

5.3 Equivalent Trains

Once the derivation of the various equations is known, one or two short-cuts can be used, and the principle of equivalent trains can be applied to this end. An equivalent train is one in which the speed of one member as shown in the basic characteristic equation is replaced by a function of the speed of one of the other members. The basic equation is $aA + sS = c\ (A + S)$ or $aN + s = c\ (N + 1)$. If, as in the Wilson first gear, the sun is the input and the cage the output, with A the reaction member, we have $s/c = N + 1$. To change the s/c ratio to obtain a different ratio we can only drive the annulus (*see* section 4.2). If we elect to drive it from the sun pinion we can write $a = ms$.

$$msN + s = c\ (N + 1)$$

$$\therefore s\ (mN + 1) = c\ (N + 1)$$

$$\therefore \frac{s}{c} = \frac{N + 1}{mN + 1} = \frac{4}{3m + 1} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (54)$$

If the new s/c ratio is 2 as before this gives

$$6m + 2 = 4 \ \therefore\ m = 1/3.$$

The A_1 annulus must, therefore, be rotated at 1/3rd of the speed of S_1. Similarly if we elect to drive it from the cage we get

$$\frac{s}{c} = N + 1 - mN \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (55)$$

When $s/c = 2$ and $N = 3$, $m = 2/3$.

This requires reduction gears of 3/1 or 3/2 and this agrees with equations (34) and (35). The result is the same as before, but in dealing with the double train as such, equation (54) can be used instead of the equation (46). This is useful when investigating the change in r_2 for a proposed change in r_1. For example, if it is

desired to change r_1 from 4 to 3·5 this means that N_1 becomes 2·5 so that r_2 becomes

$$\frac{3 \cdot 5}{\dfrac{2 \cdot 5}{3} + 1} = \frac{3 \times 3 \cdot 5}{5 \cdot 5} = 1 \cdot 91$$

The result the same as by using equation (46) but the manipulation is a little easier.

CHAPTER 6

TORQUE DISTRIBUTION

We have so far confined ourselves to the relative movements of the various members of the epicyclic train without reference to any loads being carried. When all the movements can be thus analysed, the loading can be deduced in terms of torque, since, for a given power transmitted, the torque varies inversely as the speed. It is convenient to use the symbols T_A, T_C and T_S for the torque on the annulus, cage and sun respectively. It is often considered that torque analysis is difficult, but by using common sense it will be found that this is not so.

6.1 Basic Train

Taking any basic train of ratio $A/S = N$ and neglecting friction, if the train is transmitting power and is in equilibrium, then the sum of the torques must be zero, *i.e.*

$$T_A + T_S + T_C = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (56)$$

Also the products of torque and speed must again be zero, *i.e.*

$$T_A a + T_S s + T_C c = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (57)$$

We also know from equation (16) that

$$Na + s - c\,(N + 1) = 0 \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (58)$$

Taking the case where S is fixed, A is the input member receiving torque T_A, and C the output member giving out torque $-T_C$. The negative sign to T_C should be noted. If two members are rotating in the same direction and the input torque is regarded as positive, it follows therefore that if work is being done the torque from the output is negative. The rule may be stated as: *If the torque is applied in the same sense as the rotation so as to put work in, the torque is positive. If the torque is applied in the opposite sense so as to take work out, the torque is negative.*

Since $s = 0$, from (57) and (58) we have

$T_A a = -T_C c$	and $Na = c\,(N + 1)$
i.e. $\dfrac{a}{c} = -\dfrac{T_C}{T_A}$	*i.e.* $\dfrac{a}{c} = \dfrac{N + 1}{N}$

$$\text{Hence } T_C = -\left\{\frac{N + 1}{N}\right\}.\ T_A \quad \ldots \quad \ldots \quad \ldots \quad (59)$$

Some torque T_S must be applied to the sun to prevent rotation, and by substituting (59) in (56),

$$T_A + T_S - T_A \left\{\frac{N+1}{N}\right\} = 0$$

which upon simplifying becomes:

$$T_S = \frac{T_A}{N} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (60)$$

A positive torque of value T_A/N must therefore be applied to restrain the sun.

Now consider an input torque T_C applied to the planet cage, still keeping the sun fixed. This gives, from (59),

$$T_A = -\left\{\frac{N}{N+1}\right\} T_C \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (61)$$

Substituting in (56),

$$-T_C\left\{\frac{N}{N+1}\right\} + T_S + T_C = 0;$$

or $T_S = -T_C/(N+1)$ ... ... (62)

In this case a negative torque of value $-T_C/(N+1)$ must be applied to restrain the sun.

A similar reasoning applied to the other inversions of the basic train using different reaction members will give the torque distribution for all the six cases and these are given in the following table.

BASIC TORQUE DISTRIBUTION
(one fixed member)

Input	*Output*	*Reaction*	T_S	T_C	T_A
A	C	S	$+\frac{T_A}{N}$	$-T_A\left\{\frac{N+1}{N}\right\}$	$+T_A$
C	A	S	$-\frac{T_C}{N+1}$	$+T_C$	$-T_C\left\{\frac{N}{N+1}\right\}$
S	C	A	$+T_S$	$-T_S\ (N+1)$	$+NT_S$
C	S	A	$-\frac{T_C}{N+1}$	$+T_C$	$-T_C\left\{\frac{N}{N+1}\right\}$
S	A	C	$+T_S$	$-T_S\ (N+1)$	$+NT_S$
A	S	C	$+\frac{T_A}{N}$	$-T_A\left\{\frac{N+1}{N}\right\}$	$+T_A$

Stated in general terms as in 3.2, for any gearbox where X, Y and Z are the input, output and reaction members, and x and y the input and output speeds, if $x/y = r$ and a torque T_x is applied to the input member then:—

$$T_y = -rT_x \quad \ldots \quad (63) \text{ and } T_z = (r-1)\,T_x \quad \ldots \quad \ldots \quad (64)$$

Any of the cases shown in the table can be rapidly arrived at using these values. For instance, let $X = C$, $Y = S$ and $Z = A$ and let $A/S = N$

then $s = c\,(N+1)$;

$$r = \frac{c}{s} = \frac{1}{N+1};$$

$$T_s = -\frac{T_C}{N+1} \text{ and } T_A = \left\{\frac{1}{N+1} - 1\right\} T_S = -\frac{N}{N+1} \times T_C$$

and this agrees with the table.

6.2 Coupled Trains

Coupled trains present few difficulties if approached systematically. The torque from the input is split up into two or more channels, and it is usual to find that all the members of some of the trains are rotating. An example will make the procedure clear, and since the Wilson gear has been dealt with so fully, it is convenient to analyse the torque distribution in second gear. The train is shown in Fig 16 in the form given in 4.7, general terms being used.

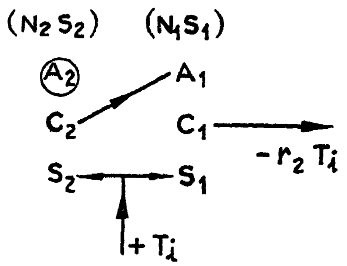


Fig 16. Torque Diagram of the Second Gear of a Wilson Gearbox

This manner of laying the train out enables the flow of torque to be examined readily and the torque paths are indicated by arrow-heads. The input torque is given the value T_i and the overall reduction ratio is r_2.

Commencing with the train containing the reaction member and using the table in 6.1 we have:—

$T_{A2} = N_2\ T_{S2}$ and $T_{C2} = -\,T_{S2}\,(N_2 + 1)$.

Also, by inspection or by definition we have for train one

$$T_{1A} = +\,T_{S2}\,(N_2+1),\ T_{C1} = -\,r_2\,T_i \text{ and } T_{S1} = T_i - T_{S2}.$$

For equilibrium the sum of these must be zero as explained in 6.1, therefore:—

$$+ T_{S2}\,(N_2 + 1) - r_2\,T_i + T_i - T_{S2} = 0;$$

hence, $T_{S2}\,N + T_{S2} - T_{S2} = r_2\,T_i - T_i$

and $\quad T_{S2} = T_i\left\{\dfrac{r_2 - 1}{N_2}\right\}$

This establishes $T_{A2} = T_i\,(r_2 - 1)$ and $T_{C2} = -\,T_i \times \dfrac{(r_2 - 1)\,(N_2 + 1)}{N_2}.$

In train one we also have:—

$$T_{A1} = +\,T_i \times \frac{(r_2 - 1)\ (N_2 + 1)}{N_2},\ T_{C1} = -\,r_2\,T_i$$

and $\quad T_{S1} = T_i - \dfrac{T_i\ (r_2 - 1)}{N_2}.$

The sign, or any change of sign, should be noted for all the terms. This can be checked by the rule given in 6.1.

As a check we can assume $T_i = 80$ lb.-ft. as being about normal in a gear of this type. Using the known values of $r_2 = 2$ and $N_2 = 2$ we get:—

$T_{A2} = +\,80$, $T_{C2} = -\,120$, $T_{S2} = +\,40$, $T_{A1} = +\,120$, $T_{C1} = -\,160$ and $T_{S1} = +\,40$. The sum of the torques in each train is zero and all the other conditions are also satisfied.

An alternative approach can be made by substituting a hypothetical basic train which gives the same ratio. Since the input torque is T_i and the output torque $-\,r_2\,T_i$ it will be obvious that the reaction torque is $(r_2 - 1)\ T_i$ and this must be applied to A_2. Reference to the table in 6.1 then enables us to write:—

$$T_{A2} = (r_2 - 1)\ T_i = N_2\ T_{S2}\ \therefore\ T_{S2} = \left\{\frac{r_2 - 1}{N_2}\right\}\ T_i$$

and $T_{C2} = -\,\dfrac{(N_2 + 1)\ (r_2 - 1)}{N_2} \times T_i.$

It then follows that

$$T_{A1} = +\,\frac{(r_2 - 1)\ (N_2 + 1)}{N_2} \times T_i \text{ and } T_{S1} = T_i - \frac{(r_2 - 1)\ T_i}{N_2}.$$

This agrees with the results of the first method and this substitution is often quicker.

6.3 Power Circulation

In some coupled trains the torque distribution may be such that power circulates *within* the train to the detriment of its efficiency.

The power so lost will depend on the efficiency of the gears and bearings, but it can be appreciable and unless one is forced to use a train of this type they are best avoided. Apart from efficiency, it means that the torque within the gear is greater on some members than it is on the output member, so necessitating a heavier gear than should be required. The third gear in the Wilson gearbox has this defect, as will be seen from examination of Fig 17, this being a diagram of the train as illustrated in Fig 11 (*b*). Inspection shows that there is a closed torque loop between second and third trains within which power is circulating. Because of this it is surprising that the train shown in Fig 11 (*a*) was not used instead.

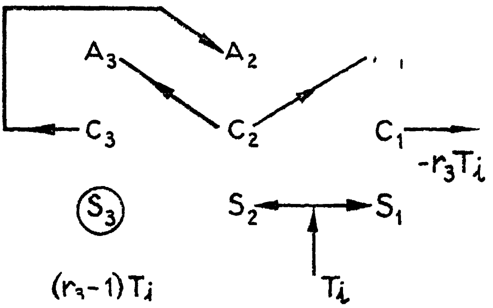


Fig 17. Use of Torque Diagram to determine Power Circulation

There may be some reason connected with the design of the direct drive clutch or with the design of the control mechanism why this train was chosen, but from the point of view of efficiency of the gears as such, the alternative train is better.

Be that as it may, it is interesting to study the torque distribution in general terms. Taking the train shown in Fig 17, let the input torque be T_i and the overall ratio r_3. The reaction torque must therefore be $(r_3 - 1)\ T_i$ and the output torque $-\ r_3\ T_i$. This gives:—

$$T_{S3} = (r_3 - 1)\ T_i \quad \ldots \quad (65); \text{ and } T_{C1} + -\ r_3\ T_i \quad \ldots \quad (66)$$

In train three, $N_3\ a_3 = c_3\ (N_3 + 1)$

hence $$\frac{a_3}{c_3} = \frac{N_3 + 1}{N_3} = r'_3,$$

therefore $T_{S3} = (r'_3 - 1)\ T_{A3}$,

leading to $$(r_3 - 1)\ T_i = \left\{\frac{N_3 + 1}{N_3} - 1\right\} T_{A3}$$

or $$T_{A3} = (r_3 - 1)\ (N_3)\ T_i \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (67)$$

This gives

$$T_{C3} = -r'_3\, T_{A3} = \frac{N_3 + 1}{N_3} \times T_{A3}$$

$$= -(N_3 + 1)\,(r_3 - 1)\, T_i \quad \ldots \quad \ldots \quad (68)$$

It follows that

$$T_{A2} = -T_{C3} = +(N_3 + 1)\,(r_3 - 1)\, T_i \quad \ldots \quad \ldots \quad (69)$$

Now consider trains two and three combined and find the ratio $s_2/c_2 = r'_2$. We know that

$$N_2\, a_2 + s_2 = c_2\,(N_2 + 1).$$

From r'_3 and the couplings it follows that

$$a_2 = c_2 \left\{\frac{N_3}{N_3 + 1}\right\}.$$

Hence $\dfrac{N_2\, N_3}{N_3 + 1}\, c_2 + s_2 = c_2\,(N_2 + 1),$

and $\quad s_2 = c_2 \left\{N_2 + 1 - \dfrac{N_2\, N_3}{N_3 + 1}\right\}$

or $\quad s_2 = c_2 \left\{\dfrac{N_3\, N_2 + N_3 + N_2 + 1 - N_2\, N_3}{N_3 + 1}\right\}$

$$= c_2 \left\{\frac{N_3 + N_2 + 1}{N_3 + 1}\right\}, \text{ with the ratio}$$

$$\frac{s_2}{c_2} = r'_2 = \frac{N_3 + N_2 + 1}{N_3 + 1}$$

The reaction torque on T_{S3} is still $(r_3 - 1)\, T_i$ but the input torque to the combined two trains is T_{S2}. Since the train ratio $r'_2 = \dfrac{N_3 + N_2 + 1}{N_3 + 1}$ we then have

$$(r_3 - 1)\, T_i = (r'_2 - 1)\, T_{S2}$$

$$= \frac{N_3 + N_2 + 1 - N_3 - 1}{N_3 + 1} \times T_{S2}$$

$$= \frac{N_2}{N_3 + 1} \times T_{S2}$$

Therefore, $T_{S2} = \dfrac{(N_3 + 1)\,(r_3 - 1)}{N_2} \times T_i \quad \ldots \quad \ldots \quad \ldots \quad (70)$

Also $T_{S1} = T_i - T_{S2}$.

That is $T_{S1} = T_i - \dfrac{(N_3 + 1)\,(r_3 - 1)}{N_2} \times T_i$

$$= \left\{1 - \frac{(N_3 + 1)\,(r_3 - 1)}{N_2}\right\} T_i \quad \ldots \quad \ldots \quad (71)$$

For train one we can write

$$T_{A1} - r_3 T_i + \left\{1 - \frac{(N_3 + 1)(r_3 - 1)}{N_2}\right\} T_i = 0$$

or $$T_{A1} = r_3 T_i - \left\{1 - \frac{(N_3 + 1)(r_3 - 1)}{N_2}\right\} T_i$$

$$= \left\{r_3 - 1 + \frac{(N_3 + 1)(r_3 - 1)}{N_2}\right\} T_i \qquad \ldots \qquad \ldots \qquad (72)$$

It follows that

$$- T_{C2} = T_{A3} + T_{A1}$$

or $$- T_{C2} = N_3 (r_3 - 1) T_i + \left\{r_3 - 1 + \frac{(N_3 + 1)(r_3 - 1)}{N_2}\right\} T_i$$

$$= \left\{N_3 (r_3 - 1) + r_3 - 1 + \frac{(N_3 + 1)(r_3 - 1)}{N_2}\right\} T_i$$

whence, $$T_{C2} = - \left\{(r_3 - 1)(N_3 + 1) + \frac{(N_3 + 1)(r_3 - 1)}{N_2}\right\} T_i \qquad (73)$$

These nine equations (65) to (73) give the torque distribution on all the co-axial members of the complete train in terms of T_i, and by substituting the known values from the example it will be found that the torque carried by C_2 is 25 per cent higher than the output torque.

6.4 Summary

Having been given an epicyclic gear train, or having arrived at a train to give specified ratios, the approach to torque evaluation is always the same. Knowing the overall ratio r_0 and the input torque T_i, the output torque is $- r_0 T_i$ and the reaction torque is $(r_0 - 1) T_i$. This is true no matter whether it is a simple basic train or a complicated coupled train. Nor does it matter which member is the input, output or reaction. Next take the train containing the reaction member and evaluate the torque in the remaining two members. This enables the next train to be taken, since the couplings are known, and so on through the complete train.

Avoid, if possible, trains in which power circulates within the train so as to produce power loss and an unduly heavy gear because of the torque increase. An alternative train can usually be evolved.

In a series of trains where several ratios are available, the torque distribution for *each* ratio should be evaluated separately. The torques carried by each member for each ratio should then be tabulated and the gear designed for the highest torque carried by each member. This is particularly important where power circulation occurs and the internal torque is higher than the external torque.

Finally, the results can be checked by adding all the torque values. The sum must always have zero value. This applies to each train considered separately or to coupled trains considered as a whole.

APPENDIX I

NOTATION

S = Teeth in sun wheel.
s = Revs of sun wheel.
P = Teeth in planet.
p = Revs of planet.
c = Revs of planet cage.
A = Teeth in annulus.
a = Revs of annulus.
N = ratio A/S.
R = ratio $\dfrac{S}{A+S}$.
x = input revs.
y = output revs.
r = ratio x/y
n = number of planets.

T_A = Torque on annulus.
T_C = Torque on cage.
T_S = Torque on sun.

Suffixes 1, 2, 3, etc., denote 1st, 2nd, 3rd trains and so on. Any other symbols used are defined in the text.

APPENDIX II

PRINCIPAL EQUATIONS

1.2. (1) $pP + sS = c\ (P + S)$.
(2) $aA + sS = c\ (A + S)$.
(3) $aA - pP = c\ (A - P)$.
(4) $aA - sS = 2pP = p\ (A - S)$.

3.2. (19) $z\ (1 - r) + yr - x = 0$.
(20) $a\ (1 - \mathrm{R}) + s\mathrm{R} - c = 0$.

6.1. (56) $T_A + T_C + T_S = 0$.
(63) $T_y = -r\,T_x$.
(64) $T_z = (r - 1)\ T_x$.

INDEX

INDEX—*continued*